# AFRL-RX-WP-TP-2011-4343

# EFFICIENT PROPAGATION OF UNCERTAINTY IN SIMULATIONS VIA THE PROBABILISTIC COLLOCATION METHOD (POSTPRINT)

**Jeremy.S. Knopp and Mark P. Blodgett**

**Nondestructive Evaluation Branch**
**Metals, Ceramics & Nondestructive Evaluation Division**

**John.C. Aldrin**
**Computational Tools**

**OCTOBER 2011**

**Approved for public release; distribution unlimited.**
***See additional restrictions described on inside pages***



**STINFO COPY**

**AIR FORCE RESEARCH LABORATORY**
**MATERIALS AND MANUFACTURING DIRECTORATE**
**WRIGHT-PATTERSON AIR FORCE BASE, OH 45433-7750**
**AIR FORCE MATERIEL COMMAND**
**UNITED STATES AIR FORCE**

# REPORT DOCUMENTATION PAGE

*Form Approved OMB No. 0704-0188*



**1. REPORT DATE *(DD-MM-YY)*** October 2011

**2. REPORT TYPE** Technical Paper

**3. DATES COVERED *(From - To)*** 1 September 2009 – 1 September 2010

**4. TITLE AND SUBTITLE**
EFFICIENT PROPAGATION OF UNCERTAINTY IN SIMULATIONS VIA THE PROBABILISTIC COLLOCATION METHOD (POSTPRINT)

**5a. CONTRACT NUMBER** In-house

**5b. GRANT NUMBER**

**5c. PROGRAM ELEMENT NUMBER** 63112F

**5d. PROJECT NUMBER** 3153

**5e. TASK NUMBER** 40

**5f. WORK UNIT NUMBER** 41300100

**6. AUTHOR(S)**
Jeremy.S. Knopp and Mark P. Blodgett (AFRL/RXLP)
John.C. Aldrin (Computational Tools)

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
Nondestructive Evaluation Branch (AFRL/RXLP)
Metals, Ceramics & Nondestructive Evaluation Division
Materials and Manufacturing Directorate
Wright-Patterson Air Force Base, OH 45433-7750
Air Force Materiel Command, United States Air Force

Computational Tools
4275 Chatham Avenue
Gurnee, IL 60031-5807

**8. PERFORMING ORGANIZATION REPORT NUMBER** AFRL-RX-WP-TP-2011-4343

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
Air Force Research Laboratory
Materials and Manufacturing Directorate
Wright-Patterson Air Force Base, OH 45433-7750
Air Force Materiel Command
United States Air Force

**10. SPONSORING/MONITORING AGENCY ACRONYM(S)** AFRL/RXLP

**11. SPONSORING/MONITORING AGENCY REPORT NUMBER(S)** AFRL-RX-WP-TP-2011-4343

**12. DISTRIBUTION/AVAILABILITY STATEMENT**
Approved for public release; distribution unlimited.

**13. SUPPLEMENTARY NOTES**
The U.S. Government is joint author of the work and has the right to use, modify, reproduce, release, perform, display, or disclose the work. PA Case Number and clearance date: 88ABW-2010-3724, 12 Jul 2010. Published in Studies in Applied Electromagnetic and Mechanics; Electromagnetic Nondestructive Evaluation Proceedings Volume 35, 2011.  This document contains color.

**14. ABSTRACT**
Eddy current models have matured to such a degree that it is now possible to simulate realistic nondestructive inspection (NDI) scenarios. Models have been used in the design and analysis of NDI systems and to a limited extent, model-based inverse methods for Nondestructive Evaluation (NDE).The science base is also being established to quantify the reliability systems via Model-Assisted Probability of Detection (MAPOD), In realistic situations, it is more accurate to treat the input model variables as random variables rather than deterministic quantities. Typically a Monte- Carlo simulation is conducted to predict the output of a model when the inputs are random variables. This is a reasonable approach as long as computational time is not to long; however, in most applications, introducing a flaw into the model results in extensive computational time ranging from hours to days, prohibiting Monte-Carlo simulations. Even methods such as Latin-Hypercube sampling do not reduce the number of simulations enough for reasonable use.

**15. SUBJECT TERMS**
eddy current, polynomial chaos, probabilistic collocation method

| 16. SECURITY CLASSIFICATION OF: a. REPORT | b. ABSTRACT | c. THIS PAGE | 17. LIMITATION OF ABSTRACT: | 18. NUMBER OF PAGES | 19a. NAME OF RESPONSIBLE PERSON (Monitor) |
|---|---|---|---|---|---|
| Unclassified | Unclassified | Unclassified | SAR | 10 | Mark Blodgett |
| | | | | | **19b. TELEPHONE NUMBER** *(Include Area Code)* N/A |

# Efficent Propagation of Uncertainty in Simulations Via the Probabilistic Collocation Method

Jeremy S. KNOPP[a,1], John C. ALDRIN[b], and Mark P. BLODGETT[a]

[a] *Air Force Research Laboratory, USA*
[b] *Computational Tools, USA*

**Abstract**. Eddy current models have matured to such a degree that it is now possible to simulate realistic nondestructive inspection (NDI) scenarios. Models have been used in the design and analysis of NDI systems and to a limited extent, model-based inverse methods for Nondestructive Evaluation (NDE). The science base is also being established to quantify the reliability of systems via Model-Assisted Probability of Detection (MAPOD). In realistic situations, it is more accurate to treat the input model variables as random variables rather than deterministic quantities. Typically a Monte-Carlo simulation is conducted to predict the output of a model when the inputs are random variables. This is a reasonable approach as long as computational time is not too long; however, in most applications, introducing a flaw into the model results in extensive computational time ranging from hours to days, prohibiting Monte-Carlo simulations. Even methods such as Latin-Hypercube sampling do not reduce the number of simulations enough for reasonable use. This paper presents the Probabilistic Collocation Method as a non-intrusive alternative to other uncertainty propagation techniques.
**Keywords:** eddy current, Polynomial Chaos, Probabilistic Collocation Method

## Introduction

Recent model development efforts have demonstrated significant advances in the simulation of eddy current inspection scenarios [1-3]. This work is a critical

[1] *Air Force Research Laboratory (AFRL-RXLP), Wright-Patterson AFB, OH 45433, USA, Email: Jeremy.Knopp@WPAFB.AF.MIL*

component of a strategy to use modeling and simulation in NDE to reduce the cost and experimental burden of Probability of Detection (POD) studies, characterize damage, and optimize probe design and inspection configuration.

The authors have investigated the utility of several numerical methods for electromagnetic NDE. These methods include analytical methods, the finite difference method, the finite element method (FEM), meshless FEM, the boundary element method (BEM), and the volume integral method. In addition to these methods, hybrid methods are also being developed [2]. While incremental advances in computational efficiency are expected, the future development of model-assisted probability of detection (MAPOD) protocols and model-based inversion schemes should not depend on a revolution in computational efficiency. One technical capability necessary for the realization of MAPOD and deployable model-based inverse methods is the propagation of uncertainty in input variables. In this work, variable refers to a measurable quantity, and parameter refers to a quantity that must be estimated such as liftoff.

Stochastic numerical methods have been developed to propagate uncertainty efficiently without relying solely on advances in computational power. Significant advances in stochastic computation have been made especially in the last 20 years. This paper briefly traces the development of stochastic numerical methods pertinent to NDE modeling and simulation. The probabilistic collocation method (PCM) is introduced and applied to two eddy current forward models.

## 1. Probabilistic Collocation Method

Spectral representation of uncertainty involves decomposing a random variable into deterministic and stochastic components. Following the work of Norbert Wiener on Homogeneous Chaos [4], Cameron and Martin pointed out that any second-order functional of Brownian motion can be expressed as a mean-square convergent series in terms of infinite-dimensional Hermite polynomials in Gaussian variables [5]. Any random process with finite second-order moments, (which encompasses most physical phenomena) can be represented by a Polynomial Chaos expansion using Hermite polynomials and will converge according to the Cameron-Martin theorem. If this random process is Gaussian, the convergence is exponential. An important property of Hermite polynomials is that they are orthogonal with respect to Gaussian probability measure. In fact for Gaussian processes, optimal convergence is achieved with the use of Hermite polynomials because the weighting function is the same as the probability density function (PDF) for the Gaussian distribution. To illustrate, the definition of an orthogonal polynomial with respect to a given weighting function w(x) is given by:

$$\left(f \cdot g\right)_w = \int_a^b f(x)g(x)w(x)dx = 0 \tag{1}$$

The inner product of functions f(x) and g(x) is zero if they are orthogonal with w(x). The only polynomial functions that satisfy this for the case where the weighting function is equal to the PDF of a standard normal distribution are Hermite polynomials.

Ghanem and Spanos combined this concept with the finite element method to simulate problems in solid mechanics [6]. For non-Gaussian processes, the convergence is not optimal, but fortunately there are other orthogonal polynomial sets for other general types of random variables that converge exponentially [7].

In work mentioned thus far, the formulations are often intrusive, meaning that alternation of deterministic code is necessary for implementation. A non-intrusive method was introduced in the mid-1990's [8]. Essentially, a reduced order polynomial form of the model is derived with minimal simulations. The input values for these simulations are derived from the roots of orthogonal polynomials which are dependent only on the input distributions. Hermite polynomials are associated with Gaussian distributions as described in the preceding paragraph. The weighting function for the Uniform distribution is simply a constant on an interval bounded by [a,b]. Legendre polynomials are orthogonal with this weighting function.

## 2. Case Study 1.

Application of the Probabilistic Collocation Method is model independent, but the decision of what order and whether to include interaction terms does depend on the complexity of the mapping of input variables to the output response. The orthogonal polynomials and their associated roots are dependent on the type of input distribution and its parameters. For this case study, the team15B workshop problem presented by Steve Burke is used [9]. All simulations are conducted using VIC-3D®. The only variable that is changed in this example is the frequency, which is 500 Hz here instead of 7 kHz. In this problem, the probe is scanned along the notch and the real and imaginary components of the impedance are measured. A peak in the magnitude of the impedance is observed when the center of the coil translated along the notch at a distance of -12 mm and 12 mm away from the origin which is the center of the notch.

The objective of this exercise is to predict the probability density function (PDF) of the peaks of the real and imaginary components of the impedance when the liftoff of the coil is uniformly distributed between 1.53 mm and 2.53 mm, and the depth of the notch is normally distributed with a mean of 5 mm and standard deviation of 1 mm. In the original problem the coil liftoff is 2.03 mm and the depth of the flaw is 5 mm.

The first step is to derive orthogonal polynomials with respect to the input distributions. From now on the parameter liftoff will be denoted by 'A', and notch depth will be denoted by 'B'. For liftoff, Legendre polynomials associated with that particular distribution and parameter set are shown in equation 2. For the depth, the Hermite polynomials are shown in equation 3.

$$\begin{aligned}
L_1(A) &= A - 2.03 \\
L_2(A) &= A^2 - 4.06A + 4.0376 \\
L_3(A) &= A^3 - 6.09A^2 + 12.2127A - 8.0609 \\
L_4(A) &= A^4 - 8.12A^3 + 24.5111A^2 - 32.5917A + 16.1041 \\
L_5(A) &= A^5 - 10.15A^4 + 40.9312A^3 - 81.9626A^2 + 81.4899A - 32.1796
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&H_1(\zeta) = \zeta \qquad && H_4(\zeta) = \zeta^4 - 6\zeta^2 + 3 \\
&H_2(\zeta) = \zeta^2 - 1 && H_5(\zeta) = \zeta^5 - 10\zeta^3 + 15\zeta \\
&H_3(\zeta) = \zeta^3 - 3\zeta &&
\end{aligned} \tag{3}
$$

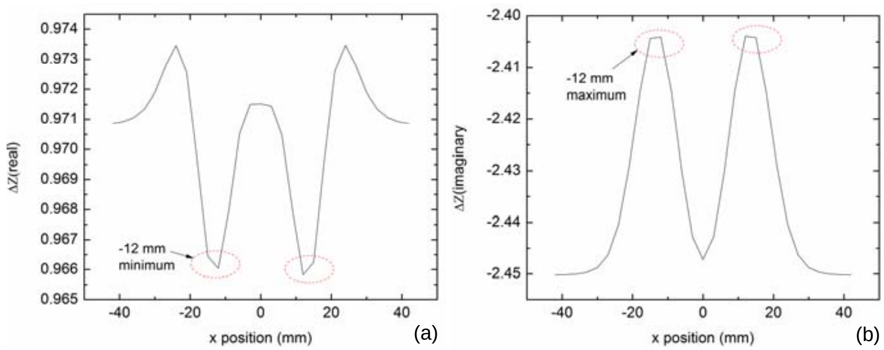


Figure 1. Change in the (a) real and (b) imaginary part of the impedance due to the plate and flaw.

Note that since the mean of the Gaussian distribution is 5, parameter B is translated by 5.

$$B = 5 + H_1(\zeta) \tag{4}$$

Next, a first order model might be used to approximate the real model. In this case, there are three unknowns $X_0$, $X_1$, and $X_2$ as shown in equation 5. Since we are interested in both the resistance and reactance, this process is done separately for each. Three simulations are required to solve for the three coefficients. Four pairs of collocation points for the inputs of the simulations are available from the roots of the 2nd order polynomials. Three pairs are selected to solve for the three unknowns. $L_1$ and $H_1$ are simply evaluated at the three pairs of collocation points. This is all that is needed for the a 1st order approximate model. Then a Monte-Carlo simulation can be conducted on this simple model to calculate the PDF of the resistance and reactance. To check the accuracy of this approximation, the next higher order collocation points are necessary. The roots of the 3rd order orthogonal polynomials are used and six simulations using the full model are conducted and compared with the approximate model. Ultimately nine simulations are necessary for construction and evaluation of the 1st order approximate model. Fortunately the same six simulations used for the error check of the 1st order model can be used to construct the 2nd order approximate model. The 4th order orthogonal polynomials are then used to check the accuracy of the 2nd order approximate model requiring eight more simulations for a total of fourteen simulations. The authors have generally found that a 3rd order approximate model with one interaction term is usually sufficient to predict output PDF's with good accuracy for forward eddy current simulations, but there may be situations where higher order approximations with more interaction terms are needed.

$$\hat{Y} = X_0 + X_1 L_1(A) + X_2 H_1(B)$$

(5)

The form of the 3rd order reduced model as described in equation 6 has eight coefficients, one of which is an interaction term.

$$\hat{Y} = X_0 + X_1 L_1(A) + X_2 H_1(\xi) + X_3 L_2(A) + X_4 H_2(\xi) + X_5 L_3(A) + X_6 H_3(\xi) + X_7 L_1(A) H_1(\xi) \quad (6)$$

To solve for the eight unknowns, eight simulations must be run. The roots of the 4th order Legendre and Hermite polynomials are used to select the input values for A and B in these simulations. This is done separately for the real and imaginary components of the impedance. Once again, everything is known except for the X coefficients. The resulting system of equations is then solved and the coefficients are listed in Table 1 for both the real and imaginary components of the impedance. Incidentally, $X_0$ is the expected value, which is one of the primary benefits of using orthogonal polynomials. As expected, the interaction term $X_7$ is close to zero in both cases.

To check the error of this third order approximation, the roots of the 5th order polynomials are used to select collocation points. Ten more simulations are needed for comparison with the reduced order model predictions. The results of the simulations and the results of the reduced model are displayed in Table 2.

Table 1. Coefficients for 3rd order approximation model.

| Model term | Real | Imaginary |
|---|---|---|
| $X_0$ | 0.9693 | -2.4132 |
| $X_1$ | -0.2209 | 0.4386 |
| $X_2$ | 0.0012 | 0.0072 |
| $X_3$ | 0.0284 | -0.0479 |
| $X_4$ | 0.0004 | -0.0011 |
| $X_5$ | -0.0050 | 0.0077 |
| $X_6$ | -0.0001 | 0.0003 |
| $X_7$ | -0.0003 | -0.0018 |

Table 2. Evaluations of original and approximation model for error calculation.

| $Y$ real | $\hat{Y}$ real | $Y$ imaginary | $\hat{Y}$ imaginary |
|---|---|---|---|
| 1.027 | 1.0270 | -2.5444 | -2.5415 |
| 1.0284 | 1.0280 | -2.5272 | -2.5295 |
| 1.0306 | 1.0305 | -2.5202 | -2.5214 |
| 0.9656 | 0.9656 | -2.4219 | -2.4195 |
| 0.9670 | 0.9666 | -2.4059 | -2.4081 |
| 0.9690 | 0.9690 | -2.3994 | -2.4007 |
| 0.9084 | 0.9084 | -2.3066 | -2.3043 |
| 0.9098 | 0.9092 | -2.2918 | -2.2937 |
| 0.9117 | 0.9116 | -2.2857 | -2.2869 |
| 0.8774 | 0.8778 | -2.2073 | -2.2053 |

The residual is defined as the difference between the full and reduced model in equation 7, and the sum of squares of the residuals is defined in equation 8. The relative sum of squares of the residuals is used to quantify the error in the approximation. The relative sum of squares of the residuals is calculated by simply dividing by the expected values as shown in equation 9. These quantities are summed over the model results for each collocation point.

$$\varepsilon = Y - \hat{Y} \tag{7}$$

$$ssr = \sqrt{\frac{\sum_i \varepsilon_i^2}{n}} \tag{8}$$

$$rssr = \frac{ssr}{E[\hat{Y}]} \tag{9}$$

The errors for the real and imaginary parts are given by: *ssr*(resistance) = $2.9428\times10^{-4}$, *rssr*(resistance) = $3.0359\times10^{-4}$, *ssr*(reactance) = 0.0020, and *rssr*(reactance) = $8.4790\times10^{-4}$.

Now there are two 3rd order polynomials with good accuracy for the real and imaginary components of the impedance. Monte Carlo simulations can now be run with ease on this 3rd order approximate model. The PDF for the real and imaginary component of the impedance is shown in Figure 2(a) and 2(b) respectively.

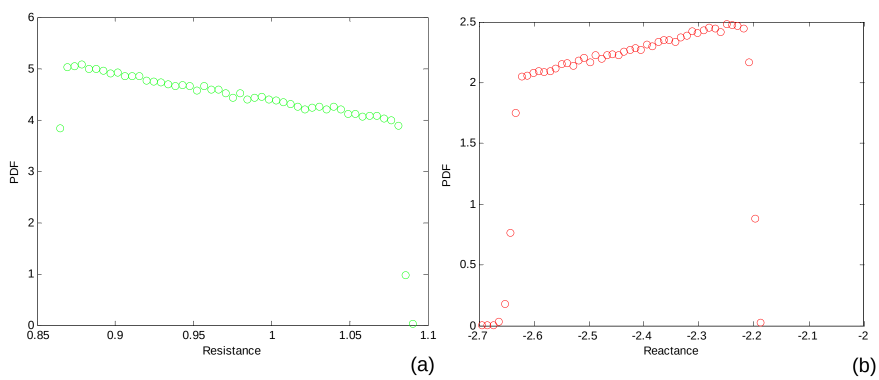


Figure 2. PDF of the (a) real and (b) imaginary component of the impedance.

## 3. Case Study 2.

A second case study is presented for the characterization of a gap between multiple plates, representing the problem of characterizing corrosion at the faying surface of a multilayer aircraft structure. A diagram of the problem is shown in Figure 3 including material properties and dimensions. In this hypothetical study, the thickness and conductivity of the layer are treated as random variables. The conductivity of the top plate is assumed to be a normal random variable with a mean value of $1.876\times10^{7}$ S/m and standard deviation of $3.384\times10^{6}$ S/m as shown in Figure 4(a). The liftoff is defined as a uniform random variable with the range of 0.55 +/- 0.05 mm as shown in Figure 2(b). VIC-3D® was used to generate the simulated results for the study. The gap between the plates was fixed at 0.3 mm and the frequency for the study was set to 4.0 kHz. Both 1st order and 2nd order PCM models were evaluated for this case requiring only 3 and 6 model calls respectively. Monte Carlo simulations were also run with 100,000 model calls used in the study.

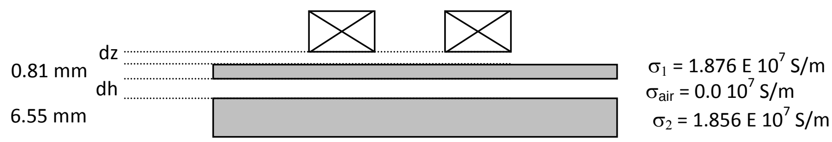


Figure 3. Diagram of case study 2 problem.

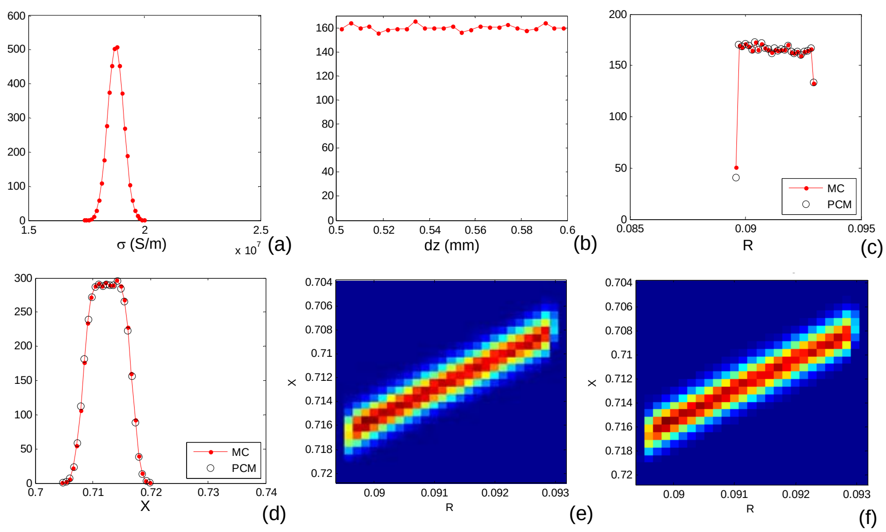


Figure 4. (a) Conductivity and (b) liftoff input distributions with ouput (c) resistance and (d) reactance output 1D distributions and (e) joint distribution calculated using PCM. (f) A joint distribution calculated using Monte Carlo simulations is provided for comparison.

Results for the 2nd order PCM model are presented in Figures 4(c)-(f). The results in Figures 4(c)-(d) show that the resistance is basically distributed uniformly and the reactance is basically distributed as a normal random variable. This relationship is expected based on the different effect of liftoff and conductivity changes in the impedance plane. A comparison of the joint distribution calculated using both PCM and Monte Carlo methods are shown in Figures 2(e) and 2(f) respectively. By observation, good agreement was achieved using PCM with Monte Carlo methods while only requiring 6 model calls here. Values for the error between the PCM and Monte Carlo simulations for the 1st order PCM model, *ssr* and *rssr*, were 43.58 and 0.3177 respectively. Error values for the 2nd order PCM model, *ssr* and *rssr*, were 15.49 and 0.1129 respectively. Thus, the 2nd order model provides a 64.5% reduction in the error between the PCM and Monte Carlo simulations with respect to the 1st order model. In conclusion, the PCM approach is highly efficient at propagating parameter variation through models with minimal model calls.

## 4. Summary

A non-intrusive approach to propagating random inputs in eddy current forward models was presented. The selection of collocation points in this method depends on the type and parameters of the input statistical distribution and not the model itself. Prediction of the PDF can be achieved with fewer simulations than traditional uncertainty propagation methods. Future work will investigate using the method with more input variables and also for inverse problems. The robustness of this method for inverse problems will likely be much more sensitive to choice of order and interaction terms.

## 5. Acknowledgements

Funding was provided in part, by the Air Force Office of Scientific Research. Many thanks to Matt Cherry for reviewing the manuscript.

## References

[1] J. S. Knopp, J. C. Aldrin, and K. V. Jata, Computational Methods in Eddy Current Crack Detection at Fastener Sites in Multi-Layer Structures, *Nondestructive Testing and Evaluation*, **24** (2009), 103-120.

[2] A. Skarlatos, C.G. Pascaud, G. Pichenot, G. Cattiaux and T. Sollier, Modelling of steam generator tubes inspection in the proximity of support plates area via a coupled finite elements – volume integral method approach, in *Electromagnetic Non-Destructive Evaluation (XII), Studies in Applied Electromagnetics and Mechanics*, Y. K. Shin, H. B. Lee, and S. J. Song, (Eds.). Amsterdam, IOS Press (2009), 51-58.

[3] H. A. Sabbagh, J. C. Aldrin, R. K. Murphy, and E. H. Sabbagh, Application of Model-Based Inversion to Eddy-Current NDE of Heat-Exchanger Tubing, in *Electromagnetic Non-Destructive Evaluation (XII), Studies in Applied Electromagnetics and Mechanics,* Y. K. Shin, H. B. Lee, and S. J. Song, (Eds.). Amsterdam, IOS Press (2009), 26-33.

[4] N. Wiener, The homogeneous chaos. American Journal of Mathematics, **60** (1938), 897–936.

[5] R. H. Cameron and W. T. Martin, The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. *Annals of Mathematics*, **48** (1947), 385–392.

[6] R. G. Ghanem and P. D. Spanos, Stochastic finite elements: a spectral approach. Springer-Verlag, New York, 1991.

[7] D. Xiu and G. E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations, *SIAM J. Sci. Comput.*, **24** (2002), 619–644.

[8] M.A. Tatang, W.W. Pan, R.G. Prinn, and G.J. McRae, An efficient method for parametric uncertainty analysis of numerical geophysical model, *J. Geophy. Res.*, **102** (1997), 21925–21932.

[9] www.compumag.org/jsite/images/stories/TEAM/problem15.pdf